POINCARÉ DUALITY EXERCISES

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In each of these exercises, M will be a closed, oriented smooth manifold.

(1) Suppose that M has dimension $2k$. Prove that the intersection pairing
$$i : H_k(M) \otimes H_k(M) \to \mathbb{Z}$$
and the wedge product pairing
$$\int_M - \wedge - : H^k_{\mathrm{dR}}(M) \otimes H^k_{\mathrm{dR}}(M) \to \mathbb{R}$$
are symmetric if k is even, and antisymmetric if k is odd.

(2) Use Stokes's theorem to prove that there is a well-defined pairing
$$\begin{aligned} H_k(M) \otimes H^k_{\mathrm{dR}}(M) &\to \mathbb{R} \\ [X] \otimes [\omega] &\mapsto \int_X \omega. \end{aligned}$$
You may assume that some multiple of every class in $H_k(M;\mathbb{Z})$ is represented by a smoothly embedded submanifold.

(3) Prove that a form $\alpha \in \Omega^1_c(\mathbb{R})$ is exact if and only if $\int_{\mathbb{R}} \alpha = 0$. Conclude that $H^1_c(\mathbb{R}) \cong \mathbb{R}$. More generally, prove that if $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuous, compactly supported, and positive somewhere then
$$\alpha = f\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \in \Omega^n_c(\mathbb{R}^n)$$
is closed but not exact, so that $H^n_c(\mathbb{R}^n)$ is nonzero.

(4) Fix a compactly supported function $f : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}} f\, dx = 1$, and define a map
$$\begin{aligned} e_n : \Omega^*_c(\mathbb{R}^n) &\to \Omega^*_c(\mathbb{R}^{n+1}) \\ \alpha &\mapsto \alpha \wedge f(x_{n+1})\, dx_{n+1}. \end{aligned}$$
Prove that e_n sends closed forms to closed forms and exact forms to exact forms, and hence defines for each k a map
$$(e_n)_* : H^k_c(\mathbb{R}^n) \to H^{k+1}_c(\mathbb{R}^{n+1}).$$
This map is an isomorphism (see §4 of Bott–Tu), which allows us to determine $H^*_c(\mathbb{R}^n)$.

(5) Let M have dimension n, and let $X, Y \subset M$ be transversely intersecting oriented submanifolds of dimensions $n-1$ and 1 respectively. Let $\eta_X \in \Omega^1(M)$ be the 1-form defined in lecture. Prove that $\int_Y \eta_X = i(X,Y)$. (Hint: work in explicit coordinates near a single point $p \in X \cap Y$.)

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